2. Mathematical description of noise

2.1. Probability: p

A physical quantity, can be measured

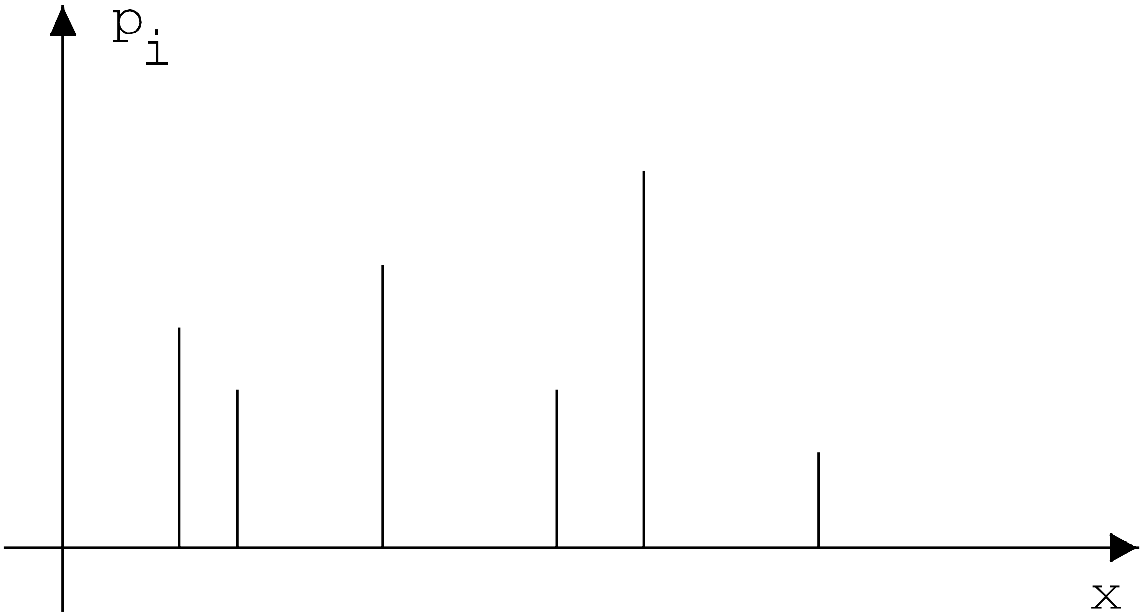
p_i of the ith event: $p_i \approx N_i/N$

N: number of experiments

N_i: number of desired events

Exactly: N->∞ *stochastic* convergence

$$\sum_i p_i = 1$$



2.2. Probability density

$p(x)\Delta x \approx N_i/N$

N: number of experiments

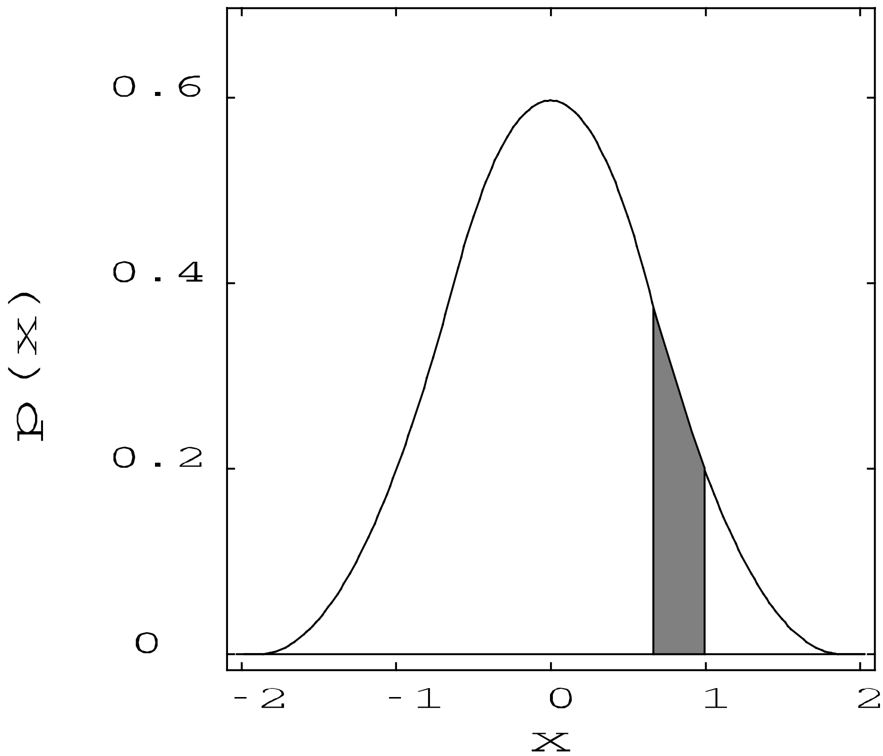
the value found N_i times in $x \pm \Delta x/2$

Exactly: $N \to \infty$ and $\Delta x \to 0$

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

Meaning: probability to find x in [x1,x2]:

$$P(x \in [x1, x2]) = \int_{x1}^{x2} p(x)\,dx$$



Complete description of a random process?

p_i or $p(x)$

2.3. Distribution function

Meaning: P(value<x) = F(x)

Derivation from p(x):

$$F(x) = \int_{-\infty}^{x} p(x')\, dx'$$

2.4. Some practical quantities

Mean value:

$$\langle x \rangle = \sum_i x_i \cdot p_i$$

$$\langle x \rangle \approx \sum_i x_i \cdot \frac{N_i}{N}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot p(x)\, dx$$

Variance, mean square:

Describes the "magnitude of fluctuations"

$$VAR(x) = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

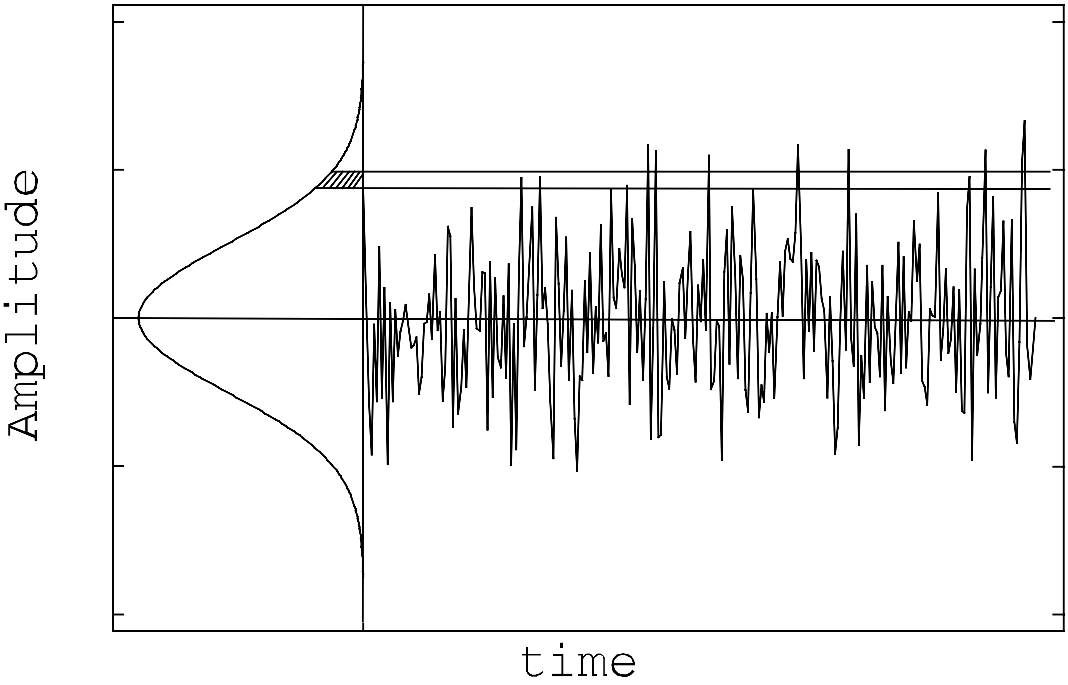
RMS or standard deviation:

$$RMS(x) = \sqrt{\langle (x - \langle x \rangle)^2 \rangle} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

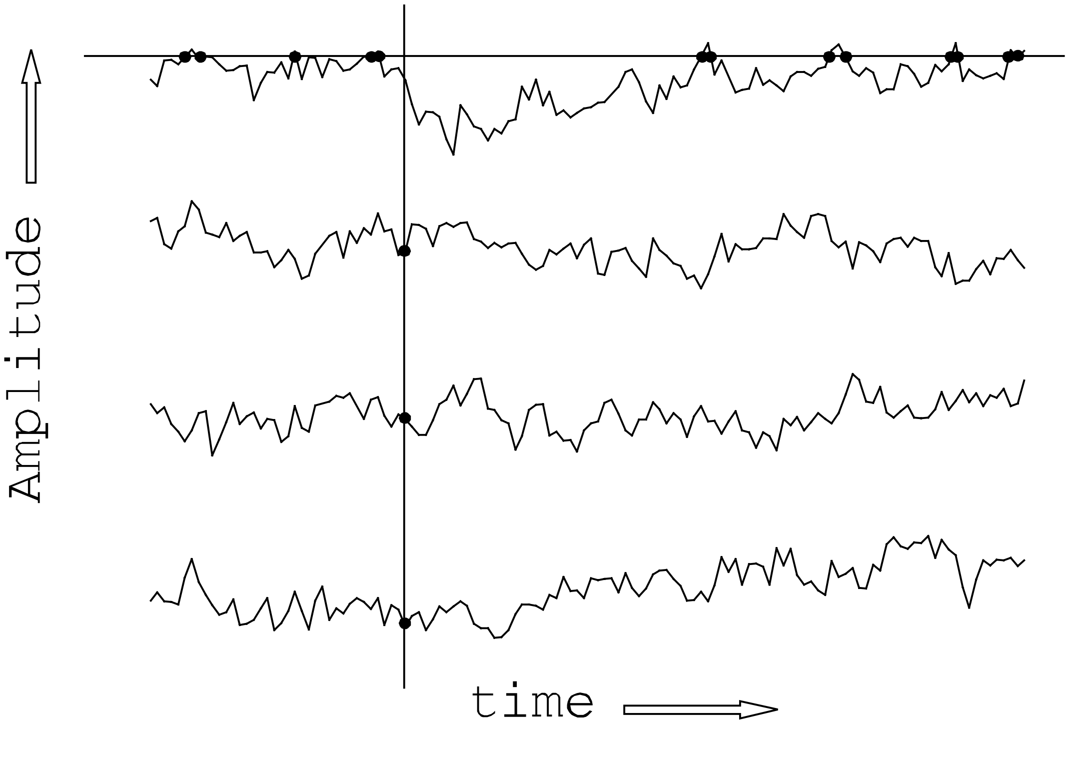
2.5. Time dependence, stationarity, ergodicity

p(x) can be time dependent: p(x,t)

A representative trajectory: x(t)



Amplitude
time



Amplitude
time

$$p(x,t)\,\Delta x \approx \frac{N_i(t)}{N}$$

$$N = \mathit{number\ of\ } x(t)\ \mathit{samples}$$

Stationary processes: p(x,t)=p(x)

Ergodic processes: ensemble <-> time avr.

$$\int_{-\infty}^{\infty} \ldots p(x)\,dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \ldots dt$$

Example: mean value

$$\int_{-\infty}^{\infty} x \cdot p(x)\,dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

2.6. Correlation functions

How to tell something about the time dependence of a random process?

- Trajectory: the most complete information
- But: too complex, simplification needed
- p(x,t), but "internal" dependencies?

2.6.1. Joint probability density

$$p(x_1,t_1,x_2,t_2)$$

2.6.2. Autocorrelation function:

Describes the "memory", internal structure of the process.

$$R_{xx}(t_1,t_2) = \langle x(t_1) \cdot x(t_2) \rangle$$

If depends on only t2-t1 (*weak stationarity*):

$$R_{xx}(\tau) = \langle x(t) \cdot x(t+\tau) \rangle$$

Ergodic process:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot x(t+\tau)\, dt$$

<u>Properties:</u>

$R_{xx}(0)$=VAR(x)=maximum

$R_{xx}(\tau)=R_{xx}(-\tau)$

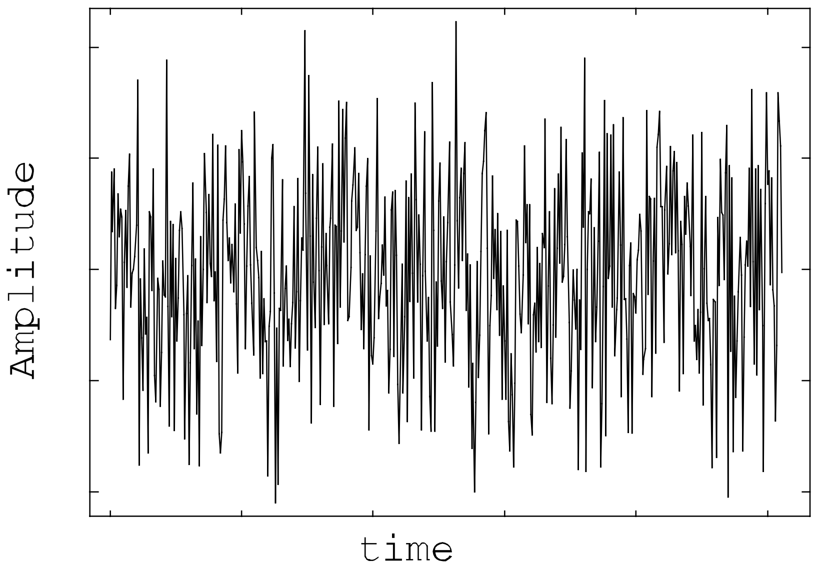
$R_{xx}(\infty)=\langle x\rangle^2$, if no periodic terms

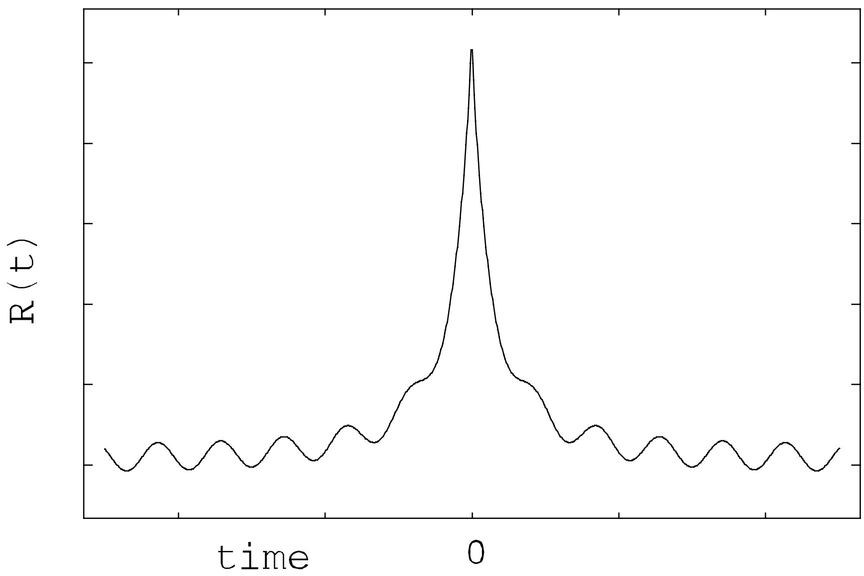
$R_{xx}(\tau)$=periodic for periodic signals

Examples:

- converges to zero for most random signals

- can be used to extract signals from noise





2.6.3. Cross correlation:

Extension of autocorrelation for two signals.

Describes time dependent relationship between random processes.

$$R_{xy}(t_1, t_2) = \langle x(t_1) \cdot y(t_2) \rangle$$

If depends on only t2-t1 (*weak stationarity*):

$$R_{xy}(\tau) = \langle x(t) \cdot y(t+\tau) \rangle$$

Ergodic process:

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot y(t+\tau) \, dt$$

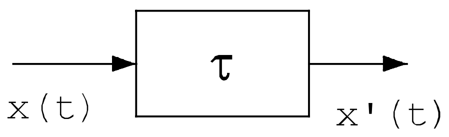
Properties:

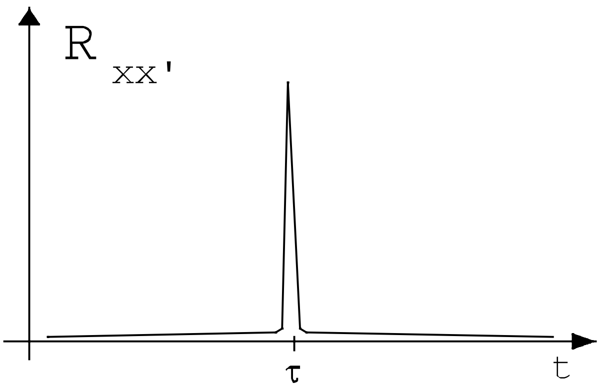
$R_{xy}(\tau)=R_{yx}(-\tau)$

$R_{xy}(\tau)=0$ for independent processes

Examples:

- can be used to extract common components from signals
- measurment of propagation of noisy signals





2.6.4. Autocorrelation of sum of multiple processes

In general:

$$R(\tau) = R_{11}(\tau) + R_{21}(\tau) + R_{12}(\tau) + R_{22}(\tau)$$

For independent processes:

$$R(\tau) = R_{11}(\tau) + R_{22}(\tau)$$

2.7. Frequency domain description

Why is it important?

decomposition to sine waves useful for e.g. description of linear systems:

- linear differential equations

- description: g(t) transfer function

$$y(t) = \int_{-\infty}^{\infty} x(t')\, g(t-t')\, dt'$$

For sine input, the output is sine also.

Since f is invariant: g(t) -> A(f),ϕ(f)

2.7.1. Fourier transform

How to calculate the components?

For periodic signals: Fourier series

$$x(t) = \sum_{k=0}^{\infty} S_k \cos(2\pi k f_o t + \varphi_k) =$$

$$\sum_{k=0}^{\infty} [A_k \cos(2\pi k f_o t) + B_i \sin(2\pi k f_o t)]$$

$$\sum_{k=-\infty}^{\infty} C_k e^{i2\pi k f_o t}$$

The coefficients:

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-i2\pi k f_o t} dt$$

$$A_k = 2\,Re(C_k)$$

$$B_k = -2\,Im(C_k)$$

Non-periodic signals:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{i2\pi ft} df$$

Properties:

- linear transform

- derivation: dx/dt -> i2πfX(f)

- integration: ∫xdt -> X(f)/i2πf

- convolution -> multiplication

$$y(t) = \int_{-\infty}^{\infty} x(t') \, g(t-t') \, dt'$$

$$Y(f) = X(f) * G(f)$$

- real x(t): $X(f)=X^*(-f)$

- $x(t)=\delta(t)$ -> $X(f)=1$

- $x(t-\tau)$ -> $e^{i2\pi\tau}X(f)$

- Parseval equality:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

2.7.2. Power density spectrum

Definition: (Wiener-Khintchine relations)

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(t)\, e^{-i2\pi f t} dt$$

$$R_{xx}(t) = \int_{-\infty}^{\infty} S_{xx}(f)\, e^{i2\pi f t} df$$

Single-sided: $S(f)=2S_{xx}(f)$, f:0..∞

Meaning: $S(f)\Delta f$ = power in $f \pm \Delta f/2$

Total power of the signal:

$$Var(x) = R_{xx}(0) = \int_{-\infty}^{\infty} S_{xx}(f)\, df = \int_{0}^{\infty} S(f)\, df$$

Energy spectrum: $2|X(f)|^2$

Cross power density spectrum:

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(t)\, e^{-i2\pi ft} dt$$

$S_{xy}(f)=0$ for independent processes

2.7.3. Power spectrum of sum of multiple processes

In general:

$$S(f) = S_{11}(f) + S_{21}(f) + S_{12}(f) + S_{22}(f)$$

For independent processes:

$$S(f) = S_{11}(f) + S_{22}(f)$$

2.7.4. Finite time analysis

Exact analysis needs infinite time for time dependent averaging, correlation and spectral analysis

Real systems: always finite time

Tradeoffs:

- cutting the signal by a window function to 0..T, e.g. w(t)=1, if tε[0,T]

$$X_T(f) = W(f) * X(f) = \int_{-\infty}^{\infty} W(f')\, X(f-f')\, df$$

- periodic expansion of the signal-> discrete spectrum at frequencies $f_n=n/T$ (Fourier-series)

The lowest frequency can be analyzed: 1/T

2.7.5. Time dependent spectral analysis

What about non-stationary processes?

- Fourier transform: integrates over time, washes out local time dependence
- E.g.:periodic signal with time dependent frequency

Solution (approx.):

- choose a time interval, and sweep in time

2.7.5.1. Wavelet analysis

Method of selection: a "window-function"

$$X(f,\tau) = \int_{-\infty}^{\infty} w(t,\tau)\, f(t)\, e^{-i2\pi f t} dt$$

Often used:

$$w(t,\tau) = \frac{f}{\sqrt{\pi}} e^{-f^2 (t-\tau)^2}$$

2.7.5.2. Windowed Fourier transform

T finite time analysis, swept over time

$$X(f,\tau) = \int_{-\infty}^{\infty} w(t,\tau)\, x(t)\, e^{-i2\pi f t} dt$$

Here the width of w(t,τ) independent of f,

e.g.:

$$w(t,\tau) = \begin{cases} 1, if\ t \in [\tau, \tau + T] \\ 0, otherwise \end{cases}$$

2.8. Classification of noises according to $p(x)$, $R_{xx}(\tau)$ and $S_{xx}(f)$

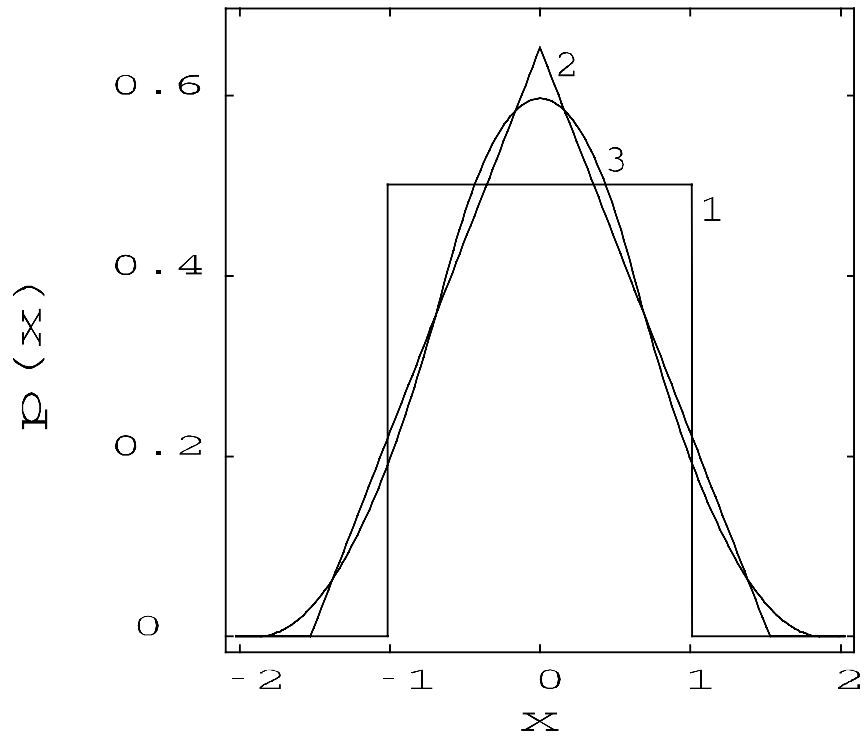
Shape of p(x)

- uniform distribution
- normal or Gaussian distribution

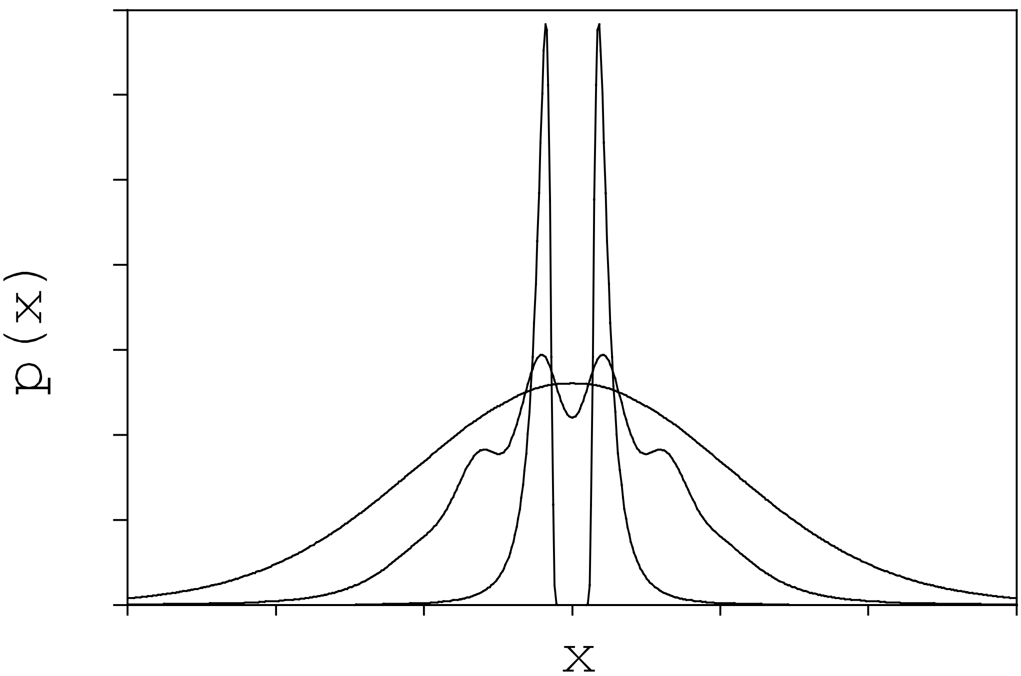
$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\langle x\rangle)^2}{2\sigma^2}}$$

- Poisson distribution
- Central limit theorem

  $y=\Sigma x_i$ -> Gaussian distribution



0.6
0.4
0.2
0
p(x)
-2
-1
0
1
2
x
1
2
3



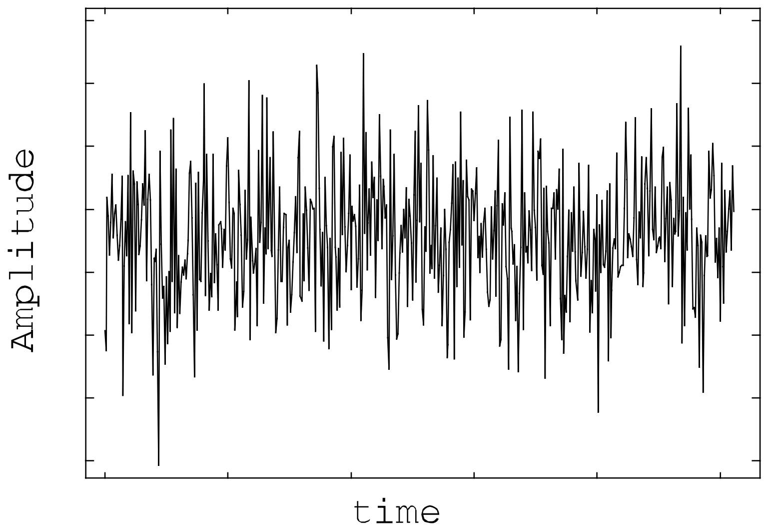
p(x)
x

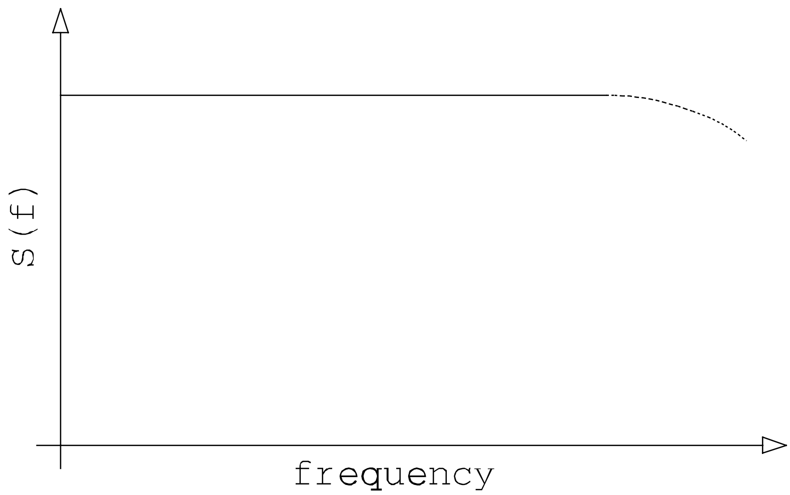
Shape of S_{xx} and R_{xx}

- White noise (uncorrelated)

  S(f)=const

  $R(\tau) \sim \delta(\tau)$

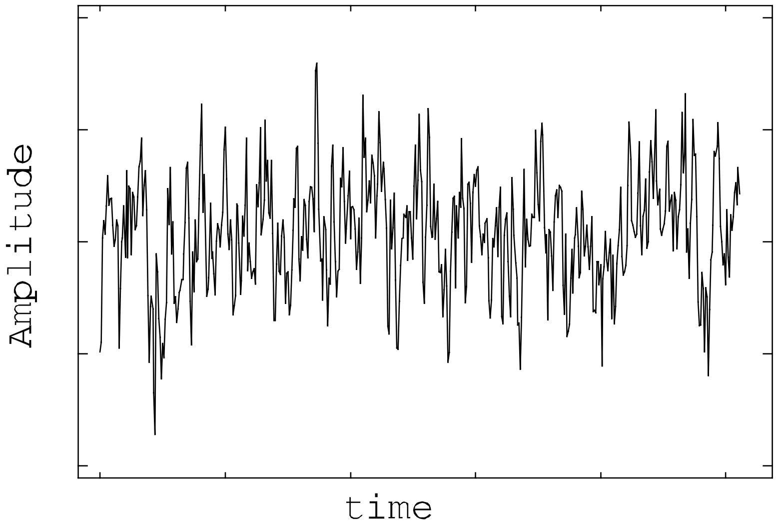


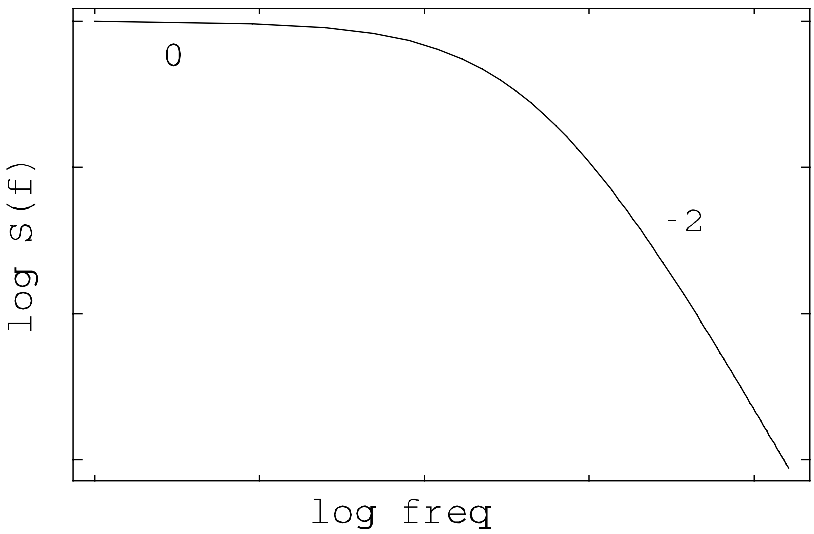


- Lorentzian noise

  $S(f) \sim 1/(1+f^2/f_o^2)$

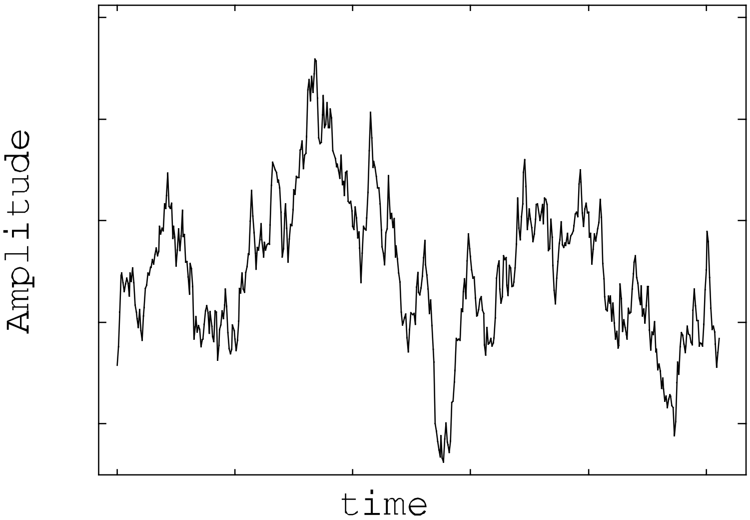
  $R(\tau) \sim \exp(-\tau/\tau_o)$, correlation time: $\tau_o=1/f_o$

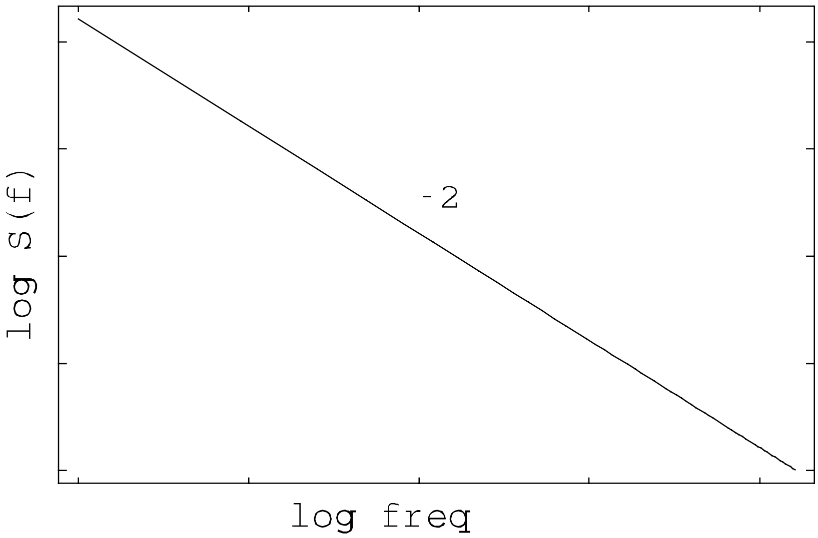




- $1/f^2$ noise

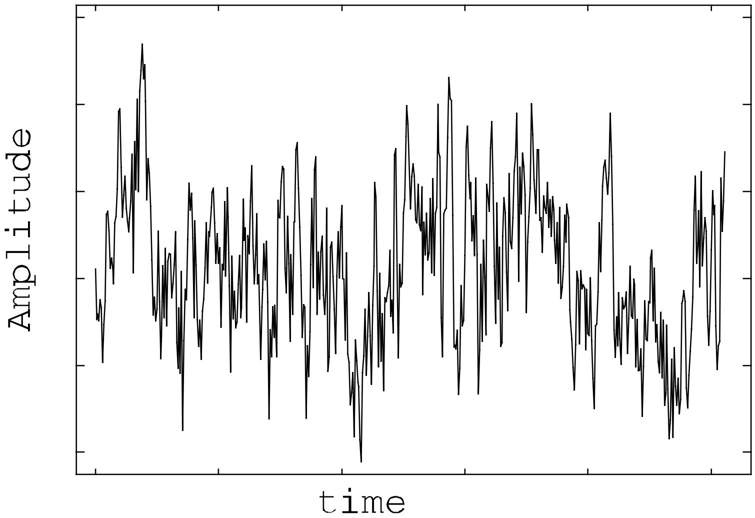
$S(f) \sim 1/f^2$, non-stationary

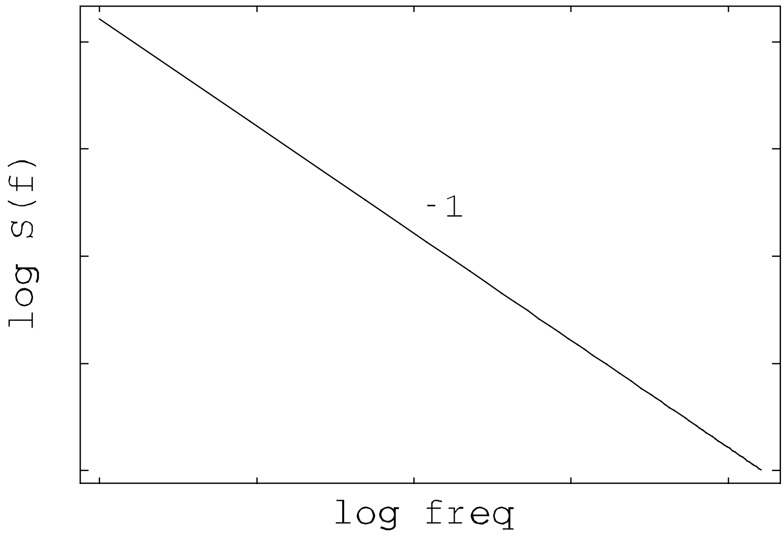




- 1/f noise (1/f$^{\kappa}$ noise, 0.8<κ<1.2)

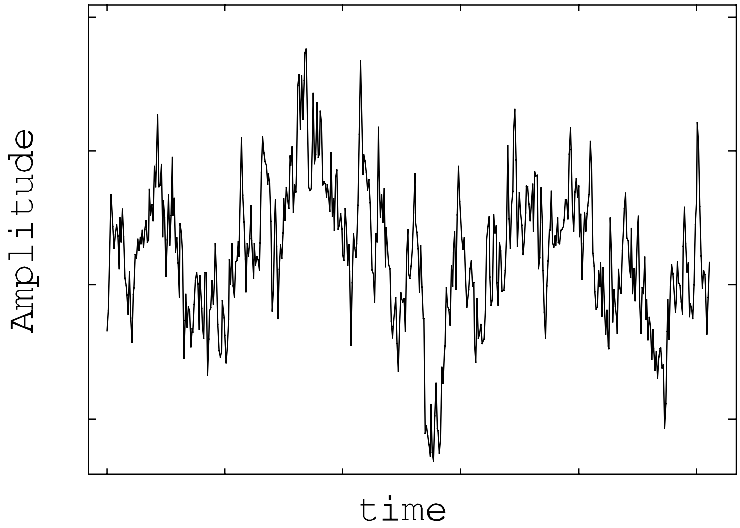
S(f) ~ 1/f$^{\kappa}$

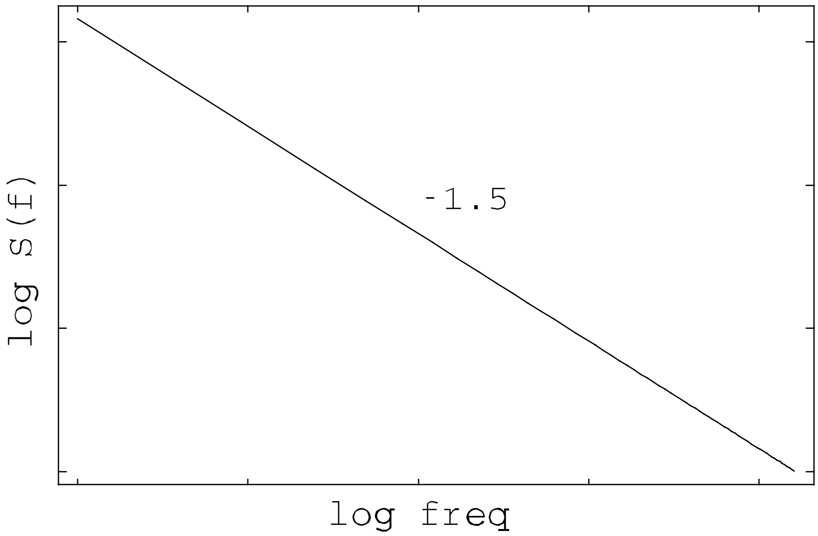


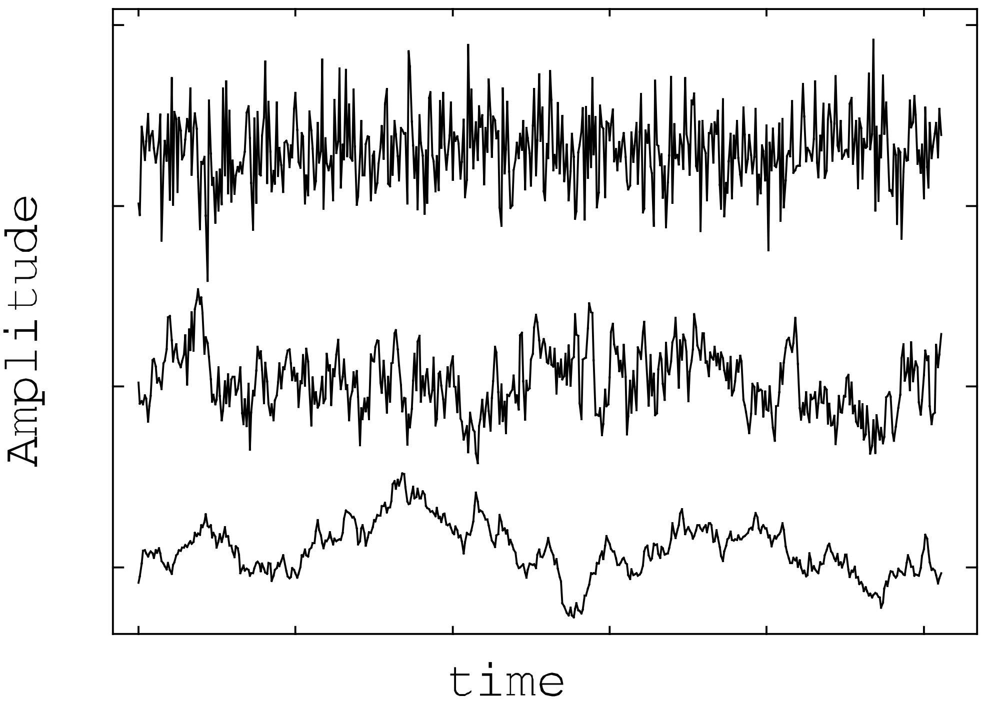


- $1/f^{1.5}$ noise

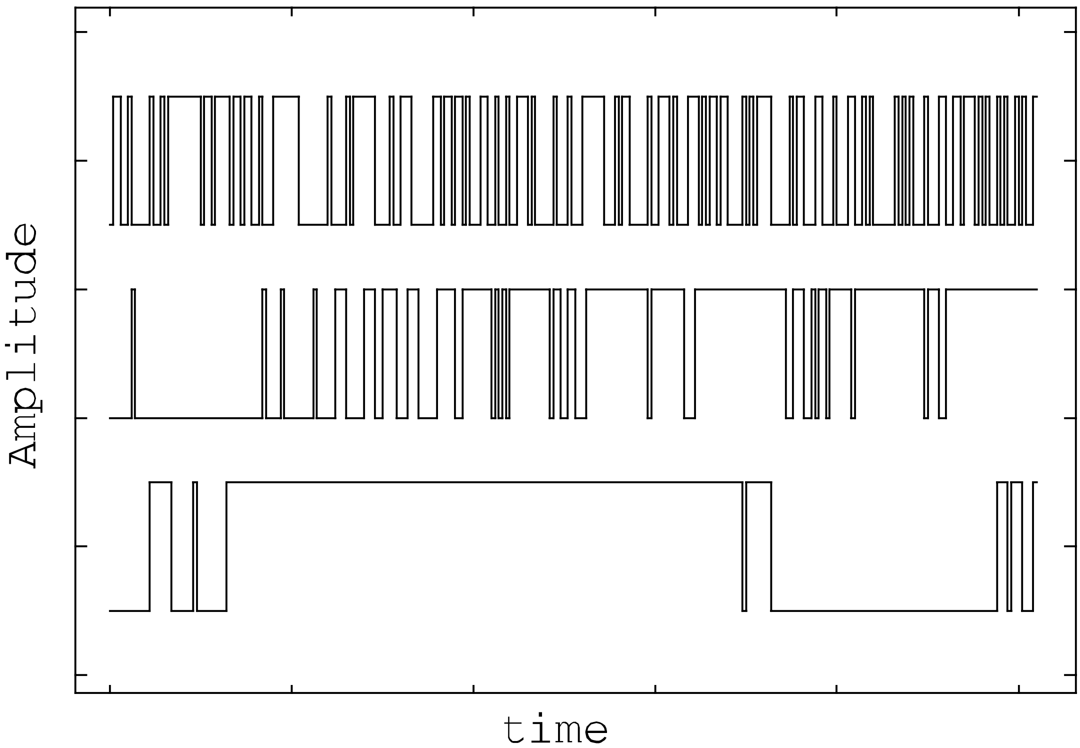
  $S(f) \sim 1/f^{1.5}$







Amplitude
time



Amplitude
time